TASHKENT UNIVERSITY OF TECHNOLOGY

Mathematics

Variant 2

CONSTRUCTOR KNOWLEDGE

Part 1

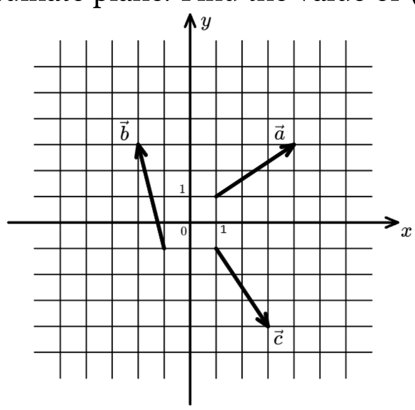
Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

1. In triangle $\triangle ABC$, it is given that $CB = 2\sqrt{2},\ AB = 3,\ \angle ABC = 45^\circ$.
Find the length of side AC.

2. A right triangular prism has a right triangular base with legs 9 and 12. The height of the prism is equal to the hypotenuse of the base. Find the surface area of the prism.

3. The vectors $\vec{a}, \vec{b}, \vec{c}$ are given in the coordinate plane. Find the value of $(\vec{a}+\vec{b})\cdot\vec{c}$.



4. The budget of city K is three times the budget of city R. City K allocated 14% of its budget to the construction of new roads. For the same purpose, city R allocated 7% of its budget. How many times greater was city K's allocation than city R's?

5. Simplify the expression

$$\left(a^{-1}-b^{-1}\right)^{-1} : \left(a^{-1}+b^{-1}\right)^{-1}, \qquad a, b \neq 0, \quad a \neq \pm b.$$

and find its value when $a = 215$ and $b = 715$.

6. Find the value of the expression:

$$\sin 435^\circ \cdot \cos(-75^\circ).$$

7. Solve the equation

$$(\sqrt{2})^x = 4.$$

8. Solve the following system:

$$\begin{cases} x+y+z=7, \\ x-y+z=-1, \\ 10x+8y-5z=47. \end{cases}$$

9. The graph of the function

$$f(x) = 2(x+5)^2 - 3$$

is reflected across the line $y = -3$, then shifted 6 units to the right and 9 units upward.
Find the coordinates of the vertex of the resulting parabola.

10. Find the absolute maximum (the greatest value) of $f(x)$ on the given interval.

$$f(x) = 3x^3 - 9x^2 + 16, \qquad [-2,\ 4].$$

Part 2

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) The flow rate of cars through a bridge checkpoint (cars per hour) is modelled by

$$F(t) = 100 + 96\cos\left(\frac{\pi(t-6)}{12}\right), \quad 0 \le t \le 12,$$

where t is the time in hours after 6 a.m.
Determine the total number of cars that pass through the checkpoint between 6 a.m. and 6 p.m. Express the answer in exact form.

12. (3 points) An ideal gas undergoes an isothermal expansion, so that its pressure P (in kPa) and volume V (in litres) satisfy the law

$$PV = 900.$$

The volume increases at a constant rate of 0.6 L/s.

(a) Express the pressure P as a function of the volume V.

(b) At the instant when $V = 30$ L, find the rate of change of the pressure.

(c) Determine the volume at which the pressure is decreasing most rapidly, given that the expansion started from an initial volume of 15 L.

13. (3 points) Solve the inequality:

$$-\ln|49x| \cdot (x^2 - 7x + 10) \cdot \left(x - \frac{31}{5}\right)^3 \cdot |x-5| \le 0.$$

14. (2 points) Two cyclists ride around a circular track of length 24 km.
They start from diametrically opposite points of the track.
If both cyclists ride clockwise, the faster cyclist catches up with the slower cyclist after 2 hours.
If one cyclist rides clockwise and the other rides counterclockwise, they meet after 20 minutes.
Find the speeds of the cyclists.

TASHKENT UNIVERSITY OF TECHNOLOGY | Mathematics | CONSTRUCTOR KNOWLEDGE

Variant 2

Part 1

Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

1. In triangle $\triangle ABC$, it is given that $CB = 2\sqrt{2},\ AB = 3,\ \angle ABC = 45°$.
Find the length of side AC.

Answer: $\sqrt{5}$

SOLUTION / PROOF :

By the cosine rule,

$$AC^2 = a^2 + c^2 - 2ac\cos\angle ABC.$$

We have

$$CB^2 = (2\sqrt{2})^2 = 8, \qquad AB^2 = 3^2 = 9.$$

Also,

$$\cos 45° = \frac{\sqrt{2}}{2}.$$

$$AC^2 = 8 + 9 - 2\cdot 3\cdot 2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 17 - 12.$$

Therefore,

$$AC = \sqrt{5}.$$

2. A right triangular prism has a right triangular base with legs 9 and 12. The height of the prism is equal to the hypotenuse of the base. Find the surface area of the prism.

Answer: 648

SOLUTION / PROOF :

First find the hypotenuse of the triangular base. Since the base is a right triangle with legs 9 and 12, by the Pythagorean theorem,

$$c = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15.$$

The height of the prism is equal to the hypotenuse of the base, so

$$h = 15.$$

The area of one triangular base is

$$S_{\text{base}} = \frac{1}{2}\cdot 9\cdot 12 = 54.$$

Since the prism has two congruent bases, their total area is

$$2S_{\text{base}} = 2\cdot 54 = 108.$$

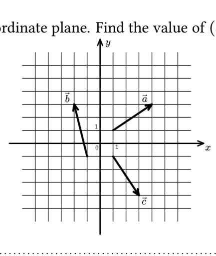
The lateral surface area is equal to the perimeter of the base multiplied by the height of the prism:

$$S_{\text{lateral}} = (9 + 12 + 15)\cdot 15 = 36\cdot 15 = 540.$$

Hence, the total surface area is

$$S = 108 + 540 = 648.$$

3. The vectors $\vec{a}, \vec{b}, \vec{c}$ are given in the coordinate plane. Find the value of $(\vec{a} + \vec{b})\cdot\vec{c}$.



Answer: −14

SOLUTION / PROOF :

Find the coordinates of the vectors:

$$\vec{a} = (4-1;\ 3-1) = (3;2), \qquad \vec{b} = (-2-(-1);\ 3-(-1)) = (-1;4), \qquad \vec{c} = (3-1;\ -4-(-1)) = (2;-3).$$

Then

$$\vec{a} + \vec{b} = (3-1;\ 2+4) = (2;6).$$

Therefore,

$$(\vec{a} + \vec{b})\cdot\vec{c} = (2;6)\cdot(2;-3) = 2\cdot 2 + 6\cdot(-3) = 4 - 18 = -14.$$

4. The budget of city K is three times the budget of city R. City K allocated 14% of its budget to the construction of new roads. For the same purpose, city R allocated 7% of its budget. How many times greater was city K's allocation than city R's?

Answer: 6

SOLUTION / PROOF :

Let the budget of city R be x. Then the budget of city K is $3x$.
City R allocated $0.07x$ to road construction, while city K allocated

$$3x\cdot 0.14 = 0.42x.$$

Therefore, the amount allocated by city K is greater by a factor of

$$\frac{0.42x}{0.07x} = 6.$$

5. Simplify the expression

$$\left(a^{-1} - b^{-1}\right)^{-1} : \left(a^{-1} + b^{-1}\right)^{-1}, \qquad a, b \neq 0, \quad a \neq \pm b.$$

and find its value when $a = 215$ and $b = 715$.

Answer: 1.86

SOLUTION / PROOF :

First simplify the first factor:

$$a^{-1} - b^{-1} = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}.$$

Therefore,

$$\left(a^{-1} - b^{-1}\right)^{-1} = \left(\frac{b-a}{ab}\right)^{-1} = \frac{ab}{b-a}.$$

Now simplify the second factor:

$$a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

Therefore,

$$\left(a^{-1} + b^{-1}\right)^{-1} = \left(\frac{a+b}{ab}\right)^{-1} = \frac{ab}{a+b}.$$

Hence,

$$\left(a^{-1} - b^{-1}\right)^{-1} : \left(a^{-1} + b^{-1}\right)^{-1} = \frac{ab}{b-a} : \frac{ab}{a+b}.$$

Dividing by a fraction means multiplying by its reciprocal:

$$\frac{ab}{b-a}\cdot\frac{a+b}{ab}.$$

Cancel ab:

$$\frac{a+b}{b-a}.$$

$$\frac{a+b}{b-a} = \frac{215 + 715}{715 - 215} = 1.86.$$

6. Find the value of the expression:

$$\sin 435°\cdot\cos(-75°).$$

Answer: 1/4 *or* 0.25

SOLUTION / PROOF :

First determine the value of

$$\sin 435°.$$

Since

$$435° = 360° + 75°,$$

we have

$$\sin 435° = \sin 75° = \cos 15°.$$

Now determine the value of

$$\cos(-75°).$$

Since cosine is an even function,

$$\cos(-75°) = \cos 75° = \sin 15°.$$

Then

$$\sin 435°\cdot\cos(-75°) = \cos 15°\cdot\sin 15°.$$

Therefore,

$$\sin 15°\cdot\cos 15° = \frac{2\sin 15°\cdot\cos 15°}{2} = \frac{\sin 30°}{2} = \frac{1}{4}.$$

7. Solve the equation

$$(\sqrt{2})^x = 4.$$

Answer: 4

SOLUTION / PROOF :

Since

$$4 = (\sqrt{2})^4,$$

we get

$$(\sqrt{2})^x = (\sqrt{2})^4.$$

Therefore,

$$x = 4.$$

8. Solve the following system:

$$\begin{cases} x + y + z = 7, \\ x - y + z = -1, \\ 10x + 8y - 5z = 47. \end{cases}$$

Answer: (2,4,1)

SOLUTION / PROOF :

Adding the 1st equation to the 2nd, we find

$$2x + 2z = 6,$$

so

$$x + z = 3.$$

From the first equation,

$$y = 7 - (x + z) = 7 - 3 = 4.$$

Therefore, the 3rd equation becomes

$$10x + 8\cdot 4 - 5z = 47.$$

Hence,

$$10x - 5z = 15.$$

Divide by 5:

$$2x - z = 3.$$

Now solve the system

$$\begin{cases} x + z = 3, \\ 2x - z = 3. \end{cases}$$

Adding the equations, we get

$$3x = 6,$$

so

$$x = 2.$$

Then

$$z = 3 - 2 = 1.$$

Therefore,

$$(x, y, z) = (2, 4, 1).$$

9. The graph of the function

$$f(x) = 2(x+5)^2 - 3$$

is reflected across the line $y = -3$, then shifted 6 units to the right and 9 units upward.
Find the coordinates of the vertex of the resulting parabola.

Answer: The vertex is $(1, 6)$.

SOLUTION / PROOF :

The original parabola

$$f(x) = 2(x+5)^2 - 3$$

has vertex

$$(-5, -3)$$

and opens upward.
Reflecting the graph across the horizontal line $y = -3$ does not change the vertex, since the vertex lies on this line. However, the parabola now opens downward. Its equation becomes

$$y = -2(x+5)^2 - 3.$$

Shifting the graph 6 units to the right changes the x-coordinate of the vertex:

$$-5 + 6 = 1.$$

Shifting the graph 9 units upward changes the y-coordinate:

$$-3 + 9 = 6.$$

Therefore, the resulting function is

$$y = -2(x-1)^2 + 6,$$

so its vertex is

$$\boxed{(1, 6)}.$$

10. Find the absolute maximum (the greatest value) of $f(x)$ on the given interval.

$$f(x) = 3x^3 - 9x^2 + 16, \qquad [-2,\ 4].$$

Answer: 64

SOLUTION / PROOF :

To find the absolute maximum on a closed interval, we check the endpoints and the critical points.
First find the derivative:

$$f'(x) = 9x^2 - 18x = 9x(x-2).$$

Critical points are found from

$$9x(x-2) = 0.$$

Hence,

$$x = 0 \qquad \text{or} \qquad x = 2.$$

Both critical points lie in the interval $[-2, 4]$.
Since the derivative is a quadratic function with coefficient $a > 0$, the derivative is positive on the intervals $[-2;0)\cup(2;4]$ and negative on the interval $(0;2)$. This means that there is a local maximum at $x = 0$.
So, we need to evaluate the function at the endpoint $x = 4$ and critical point $x = 0$:

$$f(0) = 16,$$

$$f(4) = 3\cdot 4^3 - 9\cdot 4^2 + 16 = 64.$$

The greatest value is

$$64.$$

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) The flow rate of cars through a bridge checkpoint (cars per hour) is modelled by

$$F(t) = 100 + 96\cos\left(\frac{\pi(t-6)}{12}\right), \qquad 0 \le t \le 12,$$

where t is the time in hours after 6 a.m.
Determine the total number of cars that pass through the checkpoint between 6 a.m. and 6 p.m. Express the answer in exact form.

Answer:

$1200 + \frac{2304}{\pi}$ *cars.*

SOLUTION / PROOF :

The total number of cars N is the integral of $F(t)$ from 0 to 12:

$$N = \int_0^{12}\left(100 + 96\cos\frac{\pi(t-6)}{12}\right)dt.$$

Compute the integral:

$$\int\left(100 + 96\cos\frac{\pi(t-6)}{12}\right)dt = 100t + \frac{1152}{\pi}\sin\frac{\pi(t-6)}{12} + C.$$

Evaluate from 0 to 12:

$$N = \left[100t + \frac{1152}{\pi}\sin\frac{\pi(t-6)}{12}\right]_0^{12}.$$

Therefore,

$$N = \left(1200 + \frac{1152}{\pi}\sin\frac{\pi}{2}\right) - \left(0 + \frac{1152}{\pi}\sin\left(-\frac{\pi}{2}\right)\right).$$

Since $\sin\frac{\pi}{2} = 1$ and $\sin\left(-\frac{\pi}{2}\right) = -1$,

$$N = 1200 + \frac{1152}{\pi} + \frac{1152}{\pi} = 1200 + \frac{2304}{\pi}.$$

Criteria:

Criterion content	Points
A correct answer is obtained with proper justification	2
The solution is generally correct, but contains a minor computational error	1.8
The method of solution is correct, but a significant mathematical error is made (incorrect transformations, an incorrectly found antiderivative, incorrectly set limits of integration, etc.)	1.2
The mathematical model is written correctly (the integral that needs to be found), but there is no further progress	1
It is shown in some way what exactly needs to be found (the integral of the flow velocity), but the mathematical model could not be written correctly and properly	0.5
Only the answer is given, without justification	0
The solution does not correspond to any of the criteria listed above	0

12. (3 points) An ideal gas undergoes an isothermal expansion, so that its pressure P (in kPa) and volume V (in litres) satisfy the law

$$PV = 900.$$

The volume increases at a constant rate of 0.6 L/s.

(a) Express the pressure P as a function of the volume V.

(b) At the instant when $V = 30$ L, find the rate of change of the pressure.

(c) Determine the volume at which the pressure is decreasing most rapidly, given that the expansion started from an initial volume of 15 L.

Answer:

(a) $P = \frac{900}{V}$, (b) $\frac{dP}{dt} = -0.6$ kPa/s, (c) $V = 15$ L.

SOLUTION / PROOF :

(a) From $PV = 900$ we immediately obtain

$$P = \frac{900}{V}.$$

(b) Differentiate both sides of $PV = 900$ with respect to time t:

$$\frac{d}{dt}(PV) = 0 \quad\Longrightarrow\quad P\frac{dV}{dt} + V\frac{dP}{dt} = 0.$$

Solving for $\frac{dP}{dt}$ gives

$$\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}.$$

Substitute $P = 900/V$:

$$\frac{dP}{dt} = -\frac{900}{V^2}\frac{dV}{dt}.$$

Now insert the known values $V = 30$ L and $\frac{dV}{dt} = 0.6$ L/s:

$$\frac{dP}{dt} = -\frac{900}{30^2}\cdot 0.6 = -\frac{900}{900}\cdot 0.6 = -0.6 \text{ kPa/s}.$$

Hence, the pressure decreases at a rate of 0.6 kPa/s.
(c) The absolute value of the rate of change of pressure is

$$\left|\frac{dP}{dt}\right| = \frac{900}{V^2}\cdot 0.6 = \frac{540}{V^2}.$$

Because V is increasing, V^2 increases, so $\left|\frac{dP}{dt}\right|$ decreases as V grows. Therefore, the pressure drops most rapidly when the volume is as small as possible.
The expansion began at $V_0 = 15$ L, so the maximum absolute rate occurs at

$$V = 15 \text{ L}.$$

At that moment,

$$\left|\frac{dP}{dt}\right| = \frac{540}{15^2} = \frac{540}{225} = 2.4 \text{ kPa/s}.$$

Criteria:

Criterion content	Points
A correct answer is obtained in part a)	0.4
A correct answer is obtained with proper justification in part b)	1.4
A correct answer is obtained with proper justification in part c)	1.2
The solution is generally correct, but contains a minor arithmetic error: 0.2 points are deducted. This criterion applies separately to each of the parts a), b), c).	–
In part b), the candidate showed that a derivative needs to be taken, but was unable to take it correctly	0.5
Only the answer is given, without justification	0
The solution does not correspond to any of the criteria listed above	0

13. (3 points) Solve the inequality:

$$-\ln|49x|\cdot(x^2 - 7x + 10)\cdot\left(x - \frac{31}{5}\right)^3\cdot|x-5| \le 0.$$

Answer: $\left[-\frac{1}{49};0\right)\cup\left(0;\frac{1}{49}\right]\cup[2;5]\cup\left[\frac{31}{5};+\infty\right)$

SOLUTION / PROOF :

Solve the inequality

$$-\ln|49x|\cdot(x^2 - 7x + 10)\cdot\left(x - \frac{31}{5}\right)^3\cdot|x-5| \le 0.$$

First, we find the domain. Since the logarithm is defined only when its argument is positive, we need

$$|49x| > 0.$$

Therefore,

$$x \neq 0.$$

Now factor the quadratic expression:

$$x^2 - 7x + 10 = (x-5)(x-2).$$

So the inequality becomes

$$-\ln|49x|\cdot(x-5)(x-2)\cdot\left(x - \frac{31}{5}\right)^3\cdot|x-5| \le 0.$$

The zeros of the factors are

$$-\ln|49x| = 0, \qquad x - 5 = 0, \qquad x - 2 = 0, \qquad x - \frac{31}{5} = 0.$$

For the logarithmic factor,

$$-\ln|49x| = 0$$

gives

$$\ln|49x| = 0.$$

Therefore,

$$|49x| = 1,$$

so

$$x = \pm\frac{1}{49}.$$

The other zeros are

$$x = 2, \qquad x = 5, \qquad x = \frac{31}{5}.$$

Thus, the important points are

$$-\frac{1}{49}, \quad 0, \quad \frac{1}{49}, \quad 2, \quad 5, \quad \frac{31}{5}.$$

The point $x = 0$ is not included in the domain.
Now consider the signs of the factors.
The factor $|x-5|$ is always non-negative. It does not change the sign of the product, but it makes the whole product equal to zero at

$$x = 5.$$

Next,

$$-\ln|49x| > 0 \quad\text{when}\quad 0 < |x| < \frac{1}{49},$$

and

$$-\ln|49x| < 0 \quad\text{when}\quad |x| > \frac{1}{49}.$$

Also,

$$(x-5)(x-2) > 0$$

outside the interval $(2, 5)$, and

$$(x-5)(x-2) < 0$$

inside the interval $(2, 5)$.
Finally,

$$\left(x - \frac{31}{5}\right)^3$$

has the same sign as

$$x - \frac{31}{5}.$$

So it is negative for

$$x < \frac{31}{5}$$

and positive for

$$x > \frac{31}{5}.$$

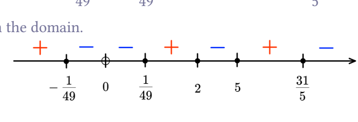
Checking the signs on the intervals determined by the important points, the product is negative on

$$\left(-\frac{1}{49}, 0\right), \qquad \left(0, \frac{1}{49}\right), \qquad (2, 5), \qquad \left(\frac{31}{5}, +\infty\right).$$

Since the inequality is non-strict, we also include the zeros of the expression:

$$x = -\frac{1}{49}, \quad x = \frac{1}{49}, \quad x = 2, \quad x = 5, \quad x = \frac{31}{5}.$$

But $x = 0$ is excluded from the domain.



Therefore, the solution is

$$x \in \left[-\frac{1}{49}, 0\right)\cup\left(0, \frac{1}{49}\right]\cup[2, 5]\cup\left[\frac{31}{5}, +\infty\right).$$

Criteria	Points
Correct answer obtained with valid justification	3
Solution carried through to completion, i.e., all solution-advancing steps were performed (domain correctly found, correct factorization, etc.), but a computational error was made	2.5
Solution carried through to completion, but a computational and/or mathematical error was made (incorrect domain, incorrect factorization, incorrect use of the interval method, incorrect zeros of the inequality, etc.)	1.5
Several solution-advancing steps correctly performed, but no further substantial progress	1
One solution-advancing step correctly performed, with no further progress or further steps containing errors	0.7
Several solution-advancing steps present, but they contain errors	0.2
Answer only, without justification	0
Solution does not meet any of the criteria listed above	0

14. (2 points) Two cyclists ride around a circular track of length 24 km.
They start from diametrically opposite points of the track.
If both cyclists ride clockwise, the faster cyclist catches up with the slower cyclist after 2 hours.
If one cyclist rides clockwise and the other rides counterclockwise, they meet after 20 minutes.
Find the speeds of the cyclists.

Answer: 21 km/h and 15 km/h

SOLUTION / PROOF :

Let the speeds of the faster and slower cyclists be v_1 and v_2 km/h.
Since the cyclists start from diametrically opposite points, the distance between them along the track is half of the track length:

$$\frac{24}{2} = 12 \text{ km}.$$

If both cyclists ride clockwise, then the faster cyclist catches the slower cyclist after 2 hours. This means the faster cyclist gains 12 km on the slower cyclist, so

$$2(v_1 - v_2) = 12.$$

Hence,

$$v_1 - v_2 = 6.$$

Now consider the case when one cyclist rides clockwise and the other rides counterclockwise. Since 20 minutes is

$$\frac{20}{60} = \frac{1}{3}$$

hours, and they move toward each other, their combined speed satisfies

$$\frac{1}{3}(v_1 + v_2) = 12.$$

Therefore,

$$v_1 + v_2 = 36.$$

We solve the system:

$$\begin{cases} v_1 + v_2 = 36, \\ v_1 - v_2 = 6. \end{cases}$$

Add the equations:

$$2v_1 = 42.$$

Therefore,

$$v_1 = 21.$$

Then

$$v_2 = 36 - 21 = 15.$$

The speeds are

$$21 \text{ km/h}$$

and

$$15 \text{ km/h}.$$

Criteria	Points
Correct mathematical model is constructed, the solution is correct, and the final answer answers the question	2
Mathematical model is correctly constructed and solved, but the answer is not exactly what the problem asks for or the final answer is incomplete	1.9
Correct equation, system, or mathematical model is constructed, but an arithmetic or algebraic error is made during solving	1.5
Correct mathematical model of the problem is constructed, or a correct sequence of actions is written, but the solution is not completed	1
Partial meaningful mathematical model is constructed: one correct meaningful equation is written, or one expression in the model/system contains an error while the work remains relevant	0.5
No relevant mathematical model is formed, or the work is unrelated to the problem	0
Answer only, without valid justification	0
Solution does not meet any of the criteria listed above	0