TASHKENT UNIVERSITY OF TECHNOLOGY

Mathematics

Variant 4

CONSTRUCTOR KNOWLEDGE

Part 1

Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

1. Find the area of a triangle with two sides of lengths 10 and 18 and an included angle of 30°.

2. A regular square pyramid $SABCD$ has base side length 2 and lateral edge length $2\sqrt{2}$. Point M is the midpoint of edge SB. Find the angle between plane AMC and the base plane.

3. Find the coordinates of the unit vector $\vec{e}$ in the direction of $\vec{a}(-4;3)$.

4. In a certain park, 60% of the trees are oaks, while the rest are cypresses. If one third of the oaks are cut down, what percentage of the remaining trees will be oaks?

5. Simplify the expression and find its value if $n = 64$.

$$\frac{2\sqrt[10]{n^9}}{\sqrt{n}\cdot\sqrt[5]{n^2}}$$

6. Find the value of the expression:

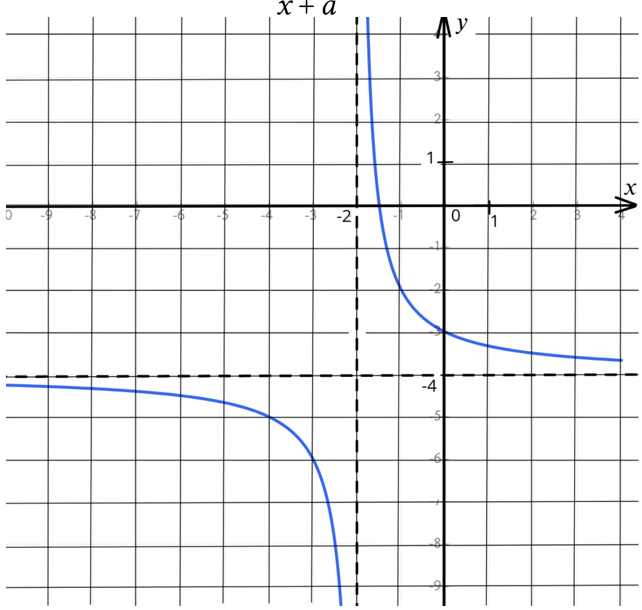
$$\log_2\left(\log_3\frac{1}{3} - \log_{\frac{1}{3}} 243\right).$$

7. Solve the equation

$$x^4 - 21x^2 = 100.$$

8. Bob's Grandpa is a very old man: his age is five times Bob's age, and four years more. When he is 90 years old, Bob will be a young man of 22. How old is Bob's Grandpa?

9. The figure below shows the graph of the function $\frac{2}{x+a} + b$. Find $a + b$.



10. Find the absolute minimum value of $f(x)$ on the given interval.

$$f(x) = \cos\left(\frac{\pi}{40}(x^2 - 2x + 4)\right), \qquad [-2, 2].$$

Part 2

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) A patient receives an injection of a certain medicine. The concentration of the medicine in the blood (in mg/L) after t hours is modelled by

$$C(t) = C_0 e^{-kt},$$

where $C_0 = 30$ mg/L and $k = 0.2\ \mathrm{h}^{-1}$.

(a) Find the rate at which the medicine concentration is decreasing at $t = 4$ hours.

(b) Calculate the half-life of the medicine, i.e. the time required for the concentration to fall to half its initial value.

(c) Determine the second derivative $C''(t)$ and explain what its sign tells about the rate of elimination over time.

12. (3 points) A particle moves along a straight line with acceleration

$$a(t) = \begin{cases} 3, & 0 \le t \le 2, \\ -5, & 2 < t \le 5. \end{cases}$$

Initially $v(0) = 2$ m/s, $x(0) = 0$.

(a) Find $v(t)$ for $0 \le t \le 5$, ensuring continuity at $t = 2$.

(b) Find the displacement over $0 \le t \le 5$.

(c) Determine when the particle changes direction.

13. (3 points) Solve the inequality:

$$\frac{x^2 - 4x + 4}{(x^2 + x - 6)\operatorname{tg}^2 x} \le 0.$$

14. (2 points) A client can choose between two bank deposits for two years.
Deposit A increases by 7% each year. Deposit B increases by 5% in the first year and by p% in the second year.
Find the smallest integer value of p for which Deposit B gives a larger final amount than Deposit A.

TASHKENT UNIVERSITY OF TECHNOLOGY | Mathematics | CONSTRUCTOR KNOWLEDGE

Variant 4

Part 1

Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

1. Find the area of a triangle with two sides of lengths 10 and 18 and an included angle of 30°.

Answer: 45

SOLUTION / PROOF :

The area of a triangle with two sides a and b and included angle γ is

$$S = \frac{1}{2}ab\sin\gamma.$$

Therefore,

$$S = \frac{1}{2}\cdot 10\cdot 18\cdot \sin 30^\circ.$$

Since

$$\sin 30^\circ = \frac{1}{2},$$

we obtain

$$S = \frac{1}{2}\cdot 10\cdot 18\cdot \frac{1}{2} = 45.$$

Thus, the area of the triangle is

$$\boxed{45}.$$

2. A regular square pyramid $SABCD$ has base side length 2 and lateral edge length $2\sqrt{2}$. Point M is the midpoint of edge SB. Find the angle between plane AMC and the base plane.

Answer: 60°

SOLUTION / PROOF :

The planes AMC and $ABCD$ intersect along line AC. Let O be the intersection point of the diagonals of the square base. Then $\angle MOB$ is the plane angle of the required dihedral angle.
Consider triangle SBD. It is isosceles with

$$SB = SD = 2.$$

The third side is the diagonal of the square base:

$$BD = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

Triangle SBD is equilateral because all its sides are equal. Hence, $\angle SDB = 60^\circ$.
Since M and O are the midpoints of sides SB and BD, respectively, segment MO is a midsegment of triangle SBD.
Hence,

$$MO \parallel SD.$$

Therefore,

$$\angle MOB = \angle SDB = 60^\circ.$$

Thus, the angle between plane AMC and the base plane is 60°.

3. Find the coordinates of the unit vector $\vec{e}$ in the direction of $\vec{a}(-4;3)$.

Answer: $(-0.8;\ 0.6)$

SOLUTION / PROOF :

Find the length of the vector $\vec{a}$:

$$|\vec{a}| = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

$$\vec{e} = \frac{1}{5}\vec{a} = \frac{1}{5}\cdot(-4;3) = (-0.8;\ 0.6).$$

4. In a certain park, 60% of the trees are oaks, while the rest are cypresses. If one third of the oaks are cut down, what percentage of the remaining trees will be oaks?

Answer: 50

SOLUTION / PROOF :

Third of the oaks account for 20% of all the trees, so the oaks that remain account for 40% of all the trees.
The remaining trees account for 80% of all the trees, and a half of them are oaks.
Therefore, after the oaks are cut down, oaks will constitute 50% of the trees in the park.

5. Simplify the expression and find its value if $n = 64$.

$$\frac{2\sqrt[10]{n^9}}{\sqrt{n}\cdot\sqrt[5]{n^2}}$$

Answer: 2

SOLUTION / PROOF :

First simplify the denominator:

$$n^{\frac{2}{5}}\cdot n^{\frac{1}{2}} = n^{\frac{2}{5}+\frac{1}{2}} = n^{\frac{9}{10}}.$$

Therefore,

$$n^{\frac{2}{5}}\cdot n^{\frac{1}{2}} = n^{\frac{9}{10}}.$$

Hence,

$$\frac{2n^{\frac{9}{10}}}{n^{\frac{9}{10}}} = 2.$$

6. Find the value of the expression:

$$\log_2\left(\log_3\frac{1}{3} - \log_{\frac{1}{3}} 243\right).$$

Answer: 2

SOLUTION / PROOF :

First evaluate

$$\log_3\frac{1}{3}.$$

Since

$$\frac{1}{3} = 3^{-1},$$

we get

$$\log_3\frac{1}{3} = -1.$$

Now evaluate

$$\log_{\frac{1}{3}} 243.$$

Since

$$243 = 3^5 \quad\text{and}\quad \frac{1}{3} = 3^{-1},$$

we get

$$\log_{\frac{1}{3}} 243 = -5.$$

Substitute these values into the original expression:

$$\log_2(-1-(-5)) = \log_2 4 = 2.$$

7. Solve the equation

$$x^4 - 21x^2 = 100.$$

Answer: $x_1 = 5, x_2 = -5$

SOLUTION / PROOF :

Let $t = x^2$. Then the equation becomes a quadratic equation in t, where $t \geq 0$:

$$t^2 - 21t - 100 = 0.$$

Find the discriminant:

$$D = (-21)^2 - 4\cdot 1\cdot(-100);$$
$$D = 441 + 400 = 841;$$
$$\sqrt{D} = 29.$$

Then

$$t_1 = \frac{21+29}{2} = 25,$$
$$t_2 = \frac{21-29}{2} = -4.$$

The second root is negative; it is not a valid solution.
Then

$$x^2 = 25;$$
$$x = 5, x = -5.$$

8. Bob's Grandpa is a very old man: his age is five times Bob's age, and four years more. When he is 90 years old, Bob will be a young man of 22. How old is Bob's Grandpa?

Answer: 84

SOLUTION / PROOF :

Let's denote Bob's current age as x and Grandpa's as y.
We know that

$$y = 5x + 4.$$

The difference between Grandpa's age and Bob's age is constant. When Grandpa is 90 and Bob is 22, the difference is

$$90 - 22 = 68.$$

Therefore,

$$y - x = 68.$$

Since $y = 5x + 4$, we get

$$5x + 4 - x = 68.$$

Hence,

$$4x + 4 = 68,$$

so

$$4x = 64, \qquad x = 16.$$

Then

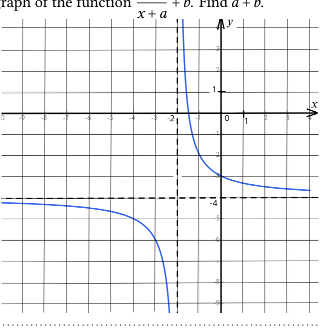
$$y = 5\cdot 16 + 4 = 84.$$

Therefore, Bob's Grandpa is

$$\boxed{84}$$

years old.

9. The figure below shows the graph of the function $\frac{2}{x+a} + b$. Find $a + b$.



Answer: −6

SOLUTION / PROOF :

The function has the form

$$y = \frac{3}{x+a} + b.$$

Its vertical asymptote is

$$x = -a.$$

From the graph, the vertical asymptote is

$$x = 2.$$

Therefore,

$$-a = 2, \qquad a = -2.$$

The horizontal asymptote is

$$y = b.$$

From the graph, the horizontal asymptote is

$$y = -4.$$

Hence,

$$b = -4.$$

Therefore,

$$a + b = -2 + (-4) = -6.$$

10. Find the absolute minimum value of $f(x)$ on the given interval.

$$f(x) = \cos\left(\frac{\pi}{40}(x^2 - 2x + 4)\right), \qquad [-2, 2].$$

Answer: $\cos\frac{3\pi}{10}$

SOLUTION / PROOF :

First find the derivative:

$$f'(x) = -\frac{\pi}{40}(2x-2)\sin\frac{\pi}{40}(x^2 - 2x + 4).$$

Its zeroes are $x = 1$ and the solutions of the quadratic equation

$$x^2 - 2x + 4 = 40n, \qquad n \in \mathbb{Z}.$$

However, on the given interval this equation has no solutions.
The other candidates for the extrema of $f(x)$ are the interval's ends. Let's check:

$$f(-2) = \cos\frac{3\pi}{10},$$
$$f(1) = \cos\frac{3\pi}{40},$$
$$f(2) = \cos\frac{\pi}{10}.$$

As we know, $\cos t$ monotonously decreases on the interval $[0, \pi/2]$.
Therefore, the minimum of $f(x)$ is

$$f(-2) = \cos\frac{3\pi}{10}.$$

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) A patient receives an injection of a certain medicine. The concentration of the medicine in the blood (in mg/L) after t hours is modelled by

$$C(t) = C_0 e^{-kt},$$

where $C_0 = 30$ mg/L and $k = 0.2\ \text{h}^{-1}$.

(a) Find the rate at which the medicine concentration is decreasing at $t = 4$ hours.

(b) Calculate the half-life of the medicine, i.e. the time required for the concentration to fall to half its initial value.

(c) Determine the second derivative $C''(t)$ and explain what its sign tells about the rate of elimination over time.

Answer: see solution

SOLUTION / PROOF :

(a)

$$C'(t) = -kC_0e^{-kt} = -0.2\cdot 30e^{-0.2t} = -6e^{-0.2t}.$$

At $t = 4$:

$$C'(4) = -6e^{-0.8} \approx -2.696\ \text{mg}\,\text{L}^{-1}\text{h}^{-1}.$$

Rounding is not necessary.
(b) Half-life $T_{1/2}$ satisfies

$$C(T_{1/2}) = \frac{1}{2}C_0.$$

Therefore,

$$30e^{-0.2T_{1/2}} = 15.$$

Hence,

$$e^{-0.2T_{1/2}} = \frac{1}{2}.$$

Taking logarithms:

$$-0.2T_{1/2} = \ln\frac{1}{2} = -\ln 2.$$

Therefore,

$$T_{1/2} = \frac{\ln 2}{0.2} = 5\ln 2 \approx 3.47\ \text{h}.$$

(c)

$$C''(t) = -6\cdot(-0.2)e^{-0.2t} = 1.2e^{-0.2t} > 0 \quad \text{for all } t.$$

The positive second derivative means the graph of $C(t)$ is concave up. Since $C'(t)$ is negative, its absolute value is decreasing; therefore the rate of elimination slows down as time passes.

Criterion content	Points
A correct answer is obtained with proper justification in part a)	0.7
A correct answer is obtained with proper justification in part b)	0.7
A correct answer is obtained with proper justification in part c)	0.6
The solution is generally correct, but contains a computational error: 0.2 points are deducted. This criterion applies separately to each of the parts a), b), c).	–
In part a), the candidate showed that a derivative needs to be taken, but was unable to take it correctly	0.2
In part b), the candidate correctly wrote down the equation, but was unable to solve it	0.2
Only the answer is given, without justification	0
The solution does not correspond to any of the criteria listed above	0

12. (3 points) A particle moves along a straight line with acceleration

$$a(t) = \begin{cases} 3, & 0 \leq t \leq 2, \\ -5, & 2 < t \leq 5. \end{cases}$$

Initially $v(0) = 2$ m/s, $x(0) = 0$.

(a) Find $v(t)$ for $0 \leq t \leq 5$, ensuring continuity at $t = 2$.

(b) Find the displacement over $0 \leq t \leq 5$.

(c) Determine when the particle changes direction.

! answer needed !

SOLUTION / PROOF :

(a) For $0 \leq t \leq 2$:

$$v(t) = \int 3\,dt = 3t + C_1.$$

Since $v(0) = 2$, we get $C_1 = 2$, so

$$v(t) = 3t + 2.$$

At $t = 2$:

$$v(2) = 3\cdot 2 + 2 = 8.$$

For $2 < t \leq 5$:

$$v(t) = \int -5\,dt = -5t + C_2.$$

Continuity at $t = 2$ gives

$$-5\cdot 2 + C_2 = 8.$$

Hence,

$$C_2 = 18.$$

Therefore,

$$v(t) = 18 - 5t.$$

Thus,

$$v(t) = \begin{cases} 3t + 2, & 0 \leq t \leq 2, \\ 18 - 5t, & 2 < t \leq 5. \end{cases}$$

(b) Displacement:

$$\int_0^2 (3t+2)\,dt + \int_2^5 (18-5t)\,dt.$$

Compute:

$$\int_0^2 (3t+2)\,dt = \left[\frac{3}{2}t^2 + 2t\right]_0^2 = 10.$$

Also,

$$\int_2^5 (18-5t)\,dt = \left[18t - \frac{5}{2}t^2\right]_2^5 = \frac{3}{2}.$$

Therefore, the displacement is

$$10 + \frac{3}{2} = \frac{23}{2}\ \text{m}.$$

(c) The particle changes direction when $v(t) = 0$.
On the first interval:

$$3t + 2 = 0$$

has no solution in $[0, 2]$.
On the second interval:

$$18 - 5t = 0.$$

Hence,

$$t = \frac{18}{5} = 3.6.$$

Criterion content	Points
A correct answer is obtained with proper justification in part a)	1
A correct answer is obtained with proper justification in part b)	1.4
A correct answer is obtained with proper justification in part c)	0.6
The solution is generally correct, but contains a minor arithmetic error: 0.2 points are deducted. This criterion applies separately to each of the parts a), b), c).	–
In part b), the candidate showed that an integral needs to be taken, but was unable to take it correctly	0.6
In part b), the candidate correctly wrote down and evaluated the integral/motion formula only on the first section of the path	0.6
In part b), the candidate correctly wrote down and evaluated the integral/motion formula only on the second section of the path	0.8
In part c), the candidate explicitly stated the condition for a change of direction, but constructed the model incorrectly	0.2
Only the answer is given, without justification	0
The solution does not correspond to any of the criteria listed above	0

13. (3 points) Solve the inequality:

$$\frac{x^2 - 4x + 4}{(x^2 + x - 6)\operatorname{tg}^2 x} \leq 0.$$

Answer: $x \in \left(-3, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\right)$

SOLUTION / PROOF :

First, factor the polynomial expressions:

$$x^2 - 4x + 4 = (x-2)^2.$$
$$x^2 + x - 6 = (x+3)(x-2).$$

So the inequality becomes

$$\frac{(x-2)^2}{(x+3)(x-2)\operatorname{tg}^2 x} \leq 0.$$

Now find the domain. The denominator must not be equal to zero, and the tangent must be defined.
We need

$$(x+3)(x-2)\operatorname{tg}^2 x \neq 0.$$

Therefore,

$$x \neq -3, \quad x \neq 2, \quad \operatorname{tg} x \neq 0.$$

Also, $\operatorname{tg} x$ must be defined, so

$$\cos x \neq 0.$$

Hence,

$$x \neq \pi k, \quad x \neq \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}.$$

On the domain, we have

$$(x-2)^2 > 0.$$

Also,

$$\operatorname{tg}^2 x > 0.$$

Therefore, the sign of the whole fraction depends only on

$$(x+3)(x-2).$$

So we need

$$(x+3)(x-2) < 0.$$

This inequality holds between the roots:

$$-3 < x < 2.$$

Now we must exclude the points where the tangent is either zero or undefined.
On the interval $(-3, 2)$, the excluded points are

$$x = -\frac{\pi}{2}, \quad x = 0, \quad x = \frac{\pi}{2}.$$

$$\begin{array}{ccccccccccc} + & & - & & - & & - & & - & & + \\ & -3 & & -\frac{\pi}{2} & & 0 & & \frac{\pi}{2} & & 2 & \end{array}$$

Therefore, the solution is

$$x \in \left(-3, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\right).$$

Criteria	Points
Correct answer obtained with valid justification	3
Solution carried through to completion, i.e., all solution-advancing steps were performed (domain correctly found, correct factorization, etc.), but a computational error was made	2.5
Solution carried through to completion, but a computational and/or mathematical error was made (incorrect domain, incorrect factorization, incorrect use of the interval method, incorrect zeros of the inequality, etc.)	1.5
Several solution-advancing steps correctly performed, but no further substantial progress	1
One solution-advancing step correctly performed, with no further progress or further steps containing errors	0.7
Several solution-advancing steps present, but they contain errors	0.2
Answer only, without justification	0
Solution does not meet any of the criteria listed above	0

14. (2 points) A client can choose between two bank deposits for two years.
Deposit A increases by 7% each year. Deposit B increases by 5% in the first year and by p% in the second year.
Find the smallest integer value of p for which Deposit B gives a larger final amount than Deposit A.

Answer: 10

SOLUTION / PROOF :

Let the initial amount be S. Since the same amount is invested in both deposits, S cancels out.
Deposit A gives

$$S\cdot 1.07^2.$$

Deposit B gives

$$S\cdot 1.05\left(1 + \frac{p}{100}\right).$$

We need

$$1.05\left(1 + \frac{p}{100}\right) > 1.07^2.$$

Therefore,

$$1 + \frac{p}{100} > \frac{1.07^2}{1.05} = \frac{1.1449}{1.05} \approx 1.09038.$$

Hence,

$$\frac{p}{100} > 0.09038,$$

so

$$p > 9.038.$$

The smallest integer value is

$$p = 10.$$

Criteria	Points
Correct mathematical model is constructed, the solution is correct, and the final answer answers the question	2
Mathematical model is correctly constructed and solved, but the answer is not exactly what the problem asks for or the final answer is incomplete	1.9
Correct equation, system, or mathematical model is constructed, but an arithmetic or algebraic error is made during solving	1.5
Correct mathematical model of the problem is constructed, or a correct sequence of actions is written, but the solution is not completed	1
Partial meaningful mathematical model is constructed: one correct meaningful equation is written, or one expression in the model/system contains an error while the work remains relevant	0.5
No relevant mathematical model is formed, or the work is unrelated to the problem	0
Answer only, without valid justification	0
Solution does not meet any of the criteria listed above	0