

Part 1

Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

- In triangle ABC , $\angle ACB = 90^\circ$ and $\angle ABC = 64^\circ$. The segment CD is a median. Find $\angle ACD$. Give your answer in degrees.
- A regular square pyramid has a base side length of 30 and a lateral edge of 25. Find its surface area.
- Given the vectors $\vec{a} = (4, -8)$ and $\vec{b} = (\lambda, 12)$, find the value of λ for which the vectors are collinear.
- Two liters of a 6% salt solution were mixed with three liters of an 11% salt solution. What is the concentration of the resulting solution?
- Simplify the expression so that the result does not contain any sums or differences.

$$\frac{\cos x + \sin 2x}{1 + \sin x - \cos 2x}.$$

- Find the value of the expression:

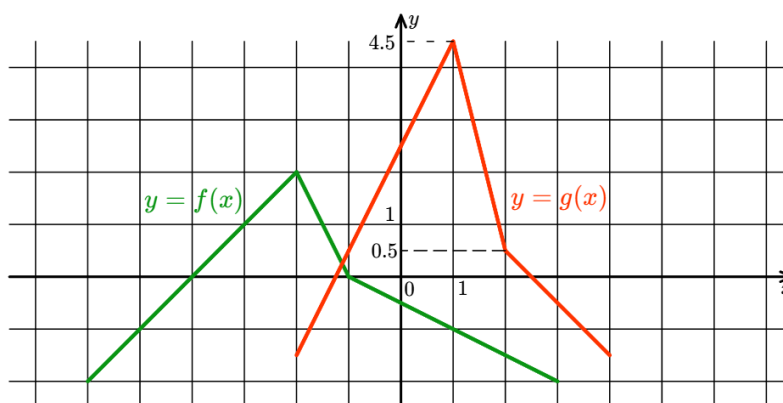
$$\frac{0.7^0 - (0.2)^{-1}}{\left(\frac{4}{3^2}\right)^{-1} \cdot \left(\frac{4}{3}\right)^2 \cdot \left(-\frac{1}{2}\right)^{-1}}.$$

- Solve the equation

$$\log_5(x-1) + \log_5(x+2) = \log_5(2x+10).$$

- Alice found out that the number of plates in her cupboard is exactly four times the number of bowls she has. But if she had three times as many bowls, she could have invited 42 guests, and each guest would have received either a plate or a bowl. How many plates does Alice have?
- The graph $y = g(x)$ is obtained from the graph $y = f(x)$ by a transformation of the form

$$g(x) = a \cdot f(x+b) + c.$$



Find the value of $a + b + c$.

10. Find the absolute minimum value of $f(x)$ on the given interval.

$$f(x) = e^{\sin 2x}, \quad \left[-\frac{\pi}{2}, 0\right]$$

Part 2

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) Oil is spilled on a calm sea and forms a circular slick of uniform thickness. The radius r of the slick increases at a constant rate of 0.3 m/min.

- (a) At the moment when the radius is 40 m, find the rate at which the area of the slick is increasing.
(b) The thickness of the slick is inversely proportional to the square of the radius:

$$h = \frac{0.02}{r^2} \text{ (metres).}$$

Find the rate of change of the thickness when $r = 40$ m.

- (c) Determine the volume of the slick when $r = 40$ m, and the rate of change of the volume at that instant. Is the volume constant?
12. (3 points) A vehicle moves along a straight test track. Its velocity v (in m/s) is given by

$$v(t) = \begin{cases} 3t, & 0 \leq t \leq 4, \\ 12 - (t - 4)^2, & 4 < t \leq 8. \end{cases}$$

- (a) Calculate the displacement of the vehicle during the first 8 seconds.
(b) Determine the total distance travelled by the vehicle in the same time interval.

13. (2 points) Solve the inequality:

$$\frac{(27 - 27x + 9x^2 - x^3)\sqrt{x-2}}{\sqrt{|x|(x^2 - 10x + 25)}} \leq 0.$$

14. (3 points) A factory produces x tons of a product per year. The yearly production cost, in thousands of dollars, is

$$3x^2 + 6x + 15.$$

The product is sold for p thousand dollars per ton. The factory wants its maximum possible yearly profit to be at least 18 thousand dollars.

Find the smallest integer value of p .

Part 1

Mathematics. Questions 1–10

Write your answer on Answer Sheet No. 1 to the right of the corresponding question number.

1. In triangle ABC , $\angle ACB = 90^\circ$ and $\angle ABC = 64^\circ$. The segment CD is a median. Find $\angle ACD$. Give your answer in degrees.

Answer: 26°

SOLUTION / PROOF :

Since the sum of the angles of a triangle is 180° ,

$$\angle BAC = 180^\circ - 90^\circ - 64^\circ = 26^\circ.$$

Since CD is a median, the point D is the midpoint of the hypotenuse AB . The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices. Therefore,

$$AD = CD.$$

Hence, triangle ACD is isosceles, so its base angles are equal:

$$\angle ACD = \angle CAD.$$

Since D lies on AB ,

$$\angle CAD = \angle CAB = 26^\circ.$$

Therefore,

$$\boxed{\angle ACD = 26^\circ}.$$

2. A regular square pyramid has a base side length of 30 and a lateral edge of 25. Find its surface area.

Answer: 2100

SOLUTION / PROOF :

Each lateral face is an isosceles triangle with side lengths 25, 25, and 30.

The altitude drawn to the base is also a median, so it divides the triangle into two congruent right triangles with hypotenuse 25 and one leg $30/2 = 15$.

Using the Pythagorean theorem, we find the height of the lateral face:

$$h = \sqrt{25^2 - 15^2} = \sqrt{625 - 225} = \sqrt{400} = 20.$$

Therefore, the area of each lateral face is

$$S_1 = \frac{1}{2} \cdot 30 \cdot 20 = 300.$$

Since the pyramid has four congruent lateral faces, their total area is

$$4S_1 = 4 \cdot 300 = 1200.$$

The area of the square base is

$$30 \cdot 30 = 900.$$

Hence, the total surface area of the pyramid is

$$S = 1200 + 900 = 2100.$$

3. Given the vectors $\vec{a} = (4, -8)$ and $\vec{b} = (\lambda, 12)$, find the value of λ for which the vectors are collinear.

Answer: -6

SOLUTION / PROOF :

Two vectors are collinear if and only if their corresponding coordinates are proportional:

$$\frac{4}{\lambda} = \frac{-8}{12},$$

$$\lambda = \frac{4 \cdot 12}{-8} = -6.$$

4. Two liters of a 6% salt solution were mixed with three liters of an 11% salt solution. What is the concentration of the resulting solution?

Answer: 9

SOLUTION / PROOF :

The amount of pure salt in the resulting solution is

$$2 \cdot 0.06 + 3 \cdot 0.11 = 0.12 + 0.33 = 0.45$$

liters.

Since the total volume of the mixture is 5 liters, the concentration of the resulting solution is

$$\frac{0.45}{5} = 0.09 = 9\%.$$

5. Simplify the expression so that the result does not contain any sums or differences.

$$\frac{\cos x + \sin 2x}{1 + \sin x - \cos 2x}$$

Answer: $\operatorname{ctg} x$

SOLUTION / PROOF :

Use the identity

$$\sin 2x = 2 \sin x \cos x.$$

Then the numerator becomes

$$\cos x + \sin 2x = \cos x + 2 \sin x \cos x = \cos x (1 + 2 \sin x).$$

For the denominator, use

$$\cos 2x = 1 - 2 \sin^2 x.$$

Hence,

$$1 + \sin x - \cos 2x = 1 + \sin x - (1 - 2 \sin^2 x) = \sin x + 2 \sin^2 x = \sin x (1 + 2 \sin x).$$

Therefore,

$$\frac{\cos x + \sin 2x}{1 + \sin x - \cos 2x} = \frac{\cos x (1 + 2 \sin x)}{\sin x (1 + 2 \sin x)} = \frac{\cos x}{\sin x} = \operatorname{ctg} x.$$

Thus, the simplified expression is

$$\boxed{\operatorname{ctg} x}.$$

6. Find the value of the expression:

$$\frac{0.7^0 - (0.2)^{-1}}{\left(\frac{4}{3^2}\right)^{-1} \cdot \left(\frac{4}{3}\right)^2 \cdot \left(-\frac{1}{2}\right)^{-1}}.$$

Answer: $\frac{1}{2}$

SOLUTION / PROOF :

First simplify the numerator:

$$0.7^0 = 1$$

and

$$(0.2)^{-1} = \left(\frac{1}{5}\right)^{-1} = 5.$$

Therefore,

$$0.7^0 - (0.2)^{-1} = 1 - 5 = -4.$$

Now simplify the denominator:

$$\left(\frac{4}{3^2}\right)^{-1} = \left(\frac{4}{9}\right)^{-1} = \frac{9}{4},$$

$$\left(\frac{4}{3}\right)^2 = \frac{16}{9},$$

and

$$\left(-\frac{1}{2}\right)^{-1} = -2.$$

Hence,

$$\left(\frac{4}{3^2}\right)^{-1} \cdot \left(\frac{4}{3}\right)^2 \cdot \left(-\frac{1}{2}\right)^{-1} = \frac{9}{4} \cdot \frac{16}{9} \cdot (-2).$$

Simplifying,

$$\frac{9}{4} \cdot \frac{16}{9} = 4,$$

so

$$4 \cdot (-2) = -8.$$

Therefore, the whole expression is

$$\frac{-4}{-8} = \frac{1}{2}.$$

7. Solve the equation

$$\log_5(x-1) + \log_5(x+2) = \log_5(2x+10).$$

Answer: 4

SOLUTION / PROOF :

Under conditions

$$x-1 > 0, \quad x+2 > 0, \quad 2x+10 > 0$$

, we obtain

$$\log_5(x-1) + \log_5(x+2) = \log_5(2x+10);$$

$$\log_5(x-1)(x+2) = \log_5(2x+10);$$

$$(x-1)(x+2) = (2x+10);$$

$$x^2 - x + 2x - 2 - 2x - 10 = 0;$$

$$x^2 - x - 12 = 0;$$

$$x_1 = -3, x_2 = 4.$$

The first root does not satisfy the given conditions and must be discarded as extraneous. Therefore, the equation has only one solution: $x = 4$.

8. Alice found out that the number of plates in her cupboard is exactly four times the number of bowls she has. But if she had three times as many bowls, she could have invited 42 guests, and each guest would have received either a plate or a bowl. How many plates does Alice have?

Answer: 24

SOLUTION / PROOF :

Let x be the number of plates and y the number of bowls.

We know that

$$x = 4y.$$

If Alice had three times as many bowls, then the total number of plates and bowls would be 42:

$$x + 3y = 42.$$

Substitute $x = 4y$ into the second equation:

$$4y + 3y = 42.$$

Hence,

$$7y = 42, \quad y = 6.$$

Therefore,

$$x = 4 \cdot 6 = 24.$$

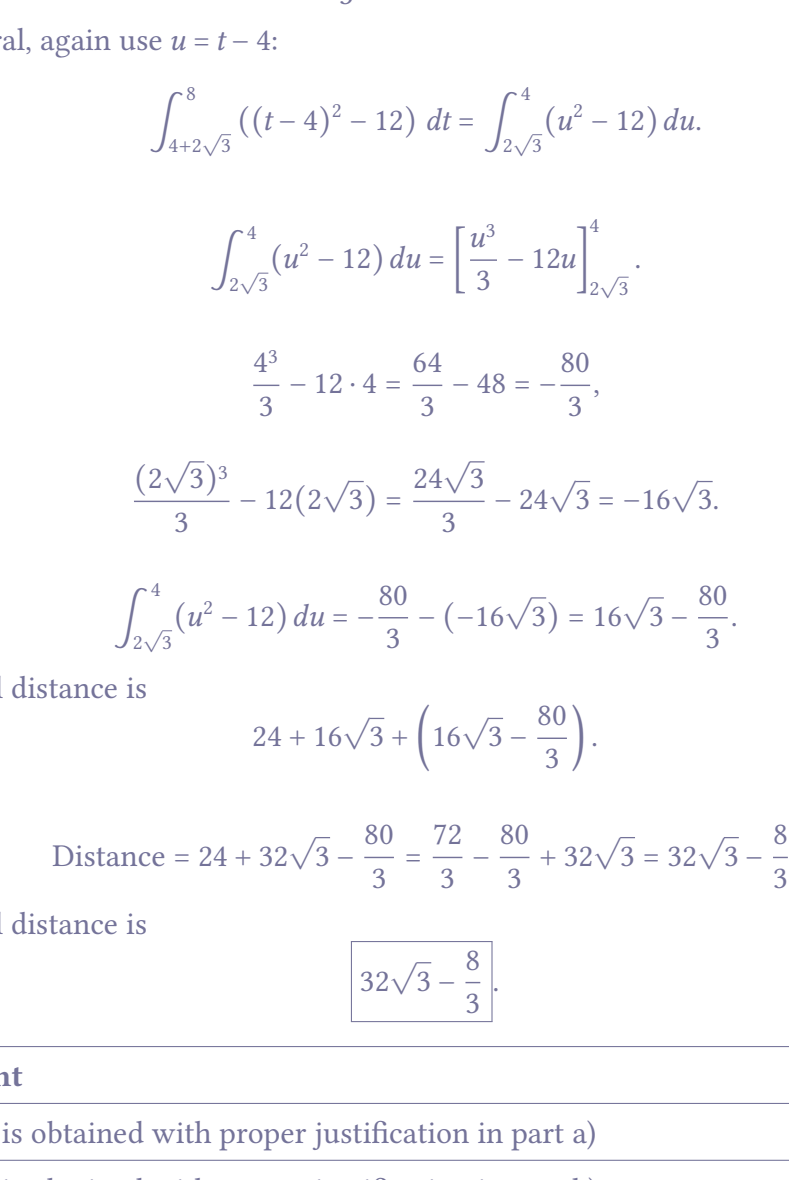
Alice has

$$\boxed{24}$$

plates.

9. The graph $y = g(x)$ is obtained from the graph $y = f(x)$ by a transformation of the form

$$g(x) = a \cdot f(x + b) + c.$$



Find the value of $a + b + c$.

Answer: $-\frac{1}{2}$

SOLUTION / PROOF :

Consider two corresponding vertices of the graph of f :

$$(-2, 2) \quad \text{and} \quad (-1, 0).$$

The corresponding vertices of the graph of g are

$$\left(1, \frac{9}{2}\right) \quad \text{and} \quad \left(2, \frac{1}{2}\right).$$

First, compare their x -coordinates. Each vertex is shifted 3 units to the right:

$$-2 \mapsto 1, \quad -1 \mapsto 2.$$

Since the transformation is written as $f(x + b)$, a shift by 3 units to the right corresponds to

$$b = -3.$$

Next, compare the vertical change between the two vertices.

For the graph of f , the change is

$$0 - 2 = -2.$$

For the graph of g , the change is

$$\frac{1}{2} - \frac{9}{2} = -4.$$

Therefore,

$$a = \frac{-4}{-2} = 2.$$

To find c , use the correspondence

$$(-1, 0) \mapsto \left(2, \frac{1}{2}\right).$$

Since

$$g(x) = 2f(x - 3) + c,$$

we get

$$\frac{1}{2} = 2 \cdot 0 + c.$$

Hence,

$$c = \frac{1}{2}.$$

Therefore,

$$g(x) = 2f(x - 3) + \frac{1}{2}.$$

Finally,

$$a + b + c = 2 - 3 + \frac{1}{2} = -\frac{1}{2}.$$

Thus,

$$\boxed{-\frac{1}{2}}.$$

10. Find the absolute minimum value of $f(x)$ on the given interval.

$$f(x) = e^{\sin 2x}, \quad \left[-\frac{\pi}{2}, 0\right]$$

Answer: $1/e$

SOLUTION / PROOF :

The derivative is $2e^{\sin 2x} \cos 2x$. Its zeroes are zeroes of $\cos 2x$, which are $x = \pm\pi/4 + \pi n$. There is only $x = -\pi/4$ on the given interval. Since the derivative change the sign from $-$ to $+$ in this point, then the absolute minimum value of $f(x)$ is $f(-\pi/4) = 1/e$.

OR

One may notice, that on the given interval $\sin 2x$ takes all its possible values, while e^x is monotonous. Therefore, maximum and minimum of $f(x)$ are taken exactly when $\sin 2x$ has maximum and minimum, which are ± 1 .

Mathematics. Questions 11–14

Use Answer Sheets No. 2 to write your solutions and answers to questions 11–14. Write your solution as completely and in as much detail as possible. Remember that even if you are unable to solve the problem completely, you may still receive some points for significant progress according to the marking criteria.

11. (2 points) Oil is spilled on a calm sea and forms a circular slick of uniform thickness. The radius r of the slick increases at a constant rate of 0.3 m/min.

- (a) At the moment when the radius is 40 m, find the rate at which the area of the slick is increasing.

- (b) The thickness of the slick is inversely proportional to the square of the radius:

$$h = \frac{0.02}{r^2} \text{ (metres).}$$

Find the rate of change of the thickness when $r = 40$ m.

- (c) Determine the volume of the slick when $r = 40$ m, and the rate of change of the volume at that instant. Is the volume constant?

Answer: see previous

SOLUTION / PROOF :

(a) The area of the slick is

$$A = \pi r^2,$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$2\pi \cdot 40 \cdot 0.3$$

$$24\pi \text{ m}^2/\text{min}.$$

(b) Thickness is

$$h = 0.02r^{-2}$$

$$\frac{dh}{dt} = 0.02 \cdot (-2) r^{-3} \frac{dr}{dt} = -0.04 r^{-3} \frac{dr}{dt}$$

At $r = 40$ m and $\frac{dr}{dt} = 0.3$ m/min:

$$\frac{dh}{dt} = -0.04 \cdot \frac{1}{40^3} \cdot 0.3 = -\frac{0.012}{64000} = -1.875 \times 10^{-7} \text{ m/min}$$

(c) The volume of the cylindrical slick is

$$V = \pi r^2 h = \pi r^2 \cdot \frac{0.02}{r^2} = 0.02\pi \text{ m}^3.$$

Since V is independent of r , it is constant. Therefore,

$$\frac{dV}{dt} = 0.$$

Thus the volume of the slick remains constant as it spreads.

Criterion content	Points
A correct answer is obtained with proper justification in part a)	0.7
A correct answer is obtained with proper justification in part b)	0.7
A correct answer is obtained with proper justification in part c)	0.6
The solution is generally correct, but contains a computational error: 0.2 points are deducted. This criterion applies separately to each of the parts a), b), c).	–
In part a), the candidate correctly wrote down the area of the slick and showed that a derivative needs to be taken, but was unable to take it correctly	0.3
In part a), the candidate only correctly wrote down the area of the slick, with no further progress	0.1
In part b), the candidate showed that a derivative needs to be taken, but was unable to take it correctly	0.3
In part c), the candidate only correctly wrote down the volume of the cylinder, with no further progress	0.2
In part c), the candidate correctly wrote down the volume of the cylinder and calculated it, with no further progress	0.3
Only the answer is given, without justification	0
The solution does not correspond to any of the criteria listed above	0

12. (3 points) A vehicle moves along a straight test track. Its velocity v (in m/s) is given by

$$v(t) = \begin{cases} 3t, & 0 \leq t \leq 4, \\ 12 - (t - 4)^2, & 4 < t \leq 8. \end{cases}$$

- (a) Calculate the displacement of the vehicle during the first 8 seconds.

- (b) Determine the total distance travelled by the vehicle in the same time interval.

Answer: (a) $\frac{152}{3}$, (b) $32\sqrt{3} - \frac{8}{3}$

SOLUTION / PROOF :

(a) The displacement is

$$s = \int_0^8 v(t) dt = \int_0^4 3t dt + \int_4^8 (12 - (t - 4)^2) dt.$$

First,

$$\int_0^4 3t dt = \left[\frac{3}{2}t^2\right]_0^4 = \frac{3}{2} \cdot 16 = 24.$$

For the second integral, substitute

$$u = t - 4, \quad du = dt.$$

Then the limits change from $t = 4$ to $t = 8$ into $u = 0$ to $u = 4$:

$$\int_4^8 (12 - (t - 4)^2) dt = \int_0^4 (12 - u^2) du.$$

Hence,

$$\int_0^4 (12 - u^2) du = \left[12u - \frac{u^3}{3}\right]_0^4 = 48 - \frac{64}{3} = \frac{80}{3}.$$

Therefore,

$$s = 24 + \frac{80}{3} = \frac{72}{3} + \frac{80}{3} = \frac{152}{3}.$$

Thus, the displacement is

$$\boxed{\frac{152}{3}}.$$

(b) The total distance is

$$\int_0^8 |v(t)| dt.$$

We need to find where the velocity changes sign. For $4 < t \leq 8$,

$$12 - (t - 4)^2 = 0.$$

Therefore,

$$(t - 4)^2 = 12.$$

Since $t > 4$, we get

$$t - 4 = 2\sqrt{3}.$$

Hence,

$$t = 4 + 2\sqrt{3}.$$

So we split the total distance into three parts:

$$\int_0^4 3t dt + \int_4^{4+2\sqrt{3}} (12 - (t - 4)^2) dt + \int_{4+2\sqrt{3}}^8 ((t - 4)^2 - 12) dt.$$

The first integral is

$$\int_0^4 3t dt = 24.$$

For the second integral, use $u = t - 4$:

$$\int_4^{4+2\sqrt{3}} (12 - (t - 4)^2) dt = \int_0^{2\sqrt{3}} (12 - u^2) du.$$

Hence,

$$\int_0^{2\sqrt{3}} (12 - u^2) du = \left[12u - \frac{u^3}{3}\right]_0^{2\sqrt{3}}.$$

Since

$$(2\sqrt{3})^3 = 24\sqrt{3},$$